# Witnessing Games in Generalized Bounded Arithmetic

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## A Proof Theoretic Goal

Investigating the power of such bounded theories of arithmetic, more specifically, finding some bounded formulas that are undecidable in these systems.

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## A Proof Theoretic Goal

Investigating the power of such bounded theories of arithmetic, more specifically, finding some bounded formulas that are undecidable in these systems.

For this purpose, it is reasonable to extract a combinatorial structure from a possible proof, hopefully to show that such combinatorial structure does not exist and hence the formula is not provable.

# Logic and Games

For such a combinatorial extraction, we need games. Interpret the formula:

$$A = \forall \vec{y_1} \leq \vec{p_1}(\vec{x}) \exists \vec{z_1} \leq \vec{q_1}(\vec{x}) \forall \vec{y_2} \leq \vec{p_2}(\vec{x}) \ldots G_A(\vec{x}, \vec{y_1}, \vec{z_1}, \vec{y_2}, \vec{z_2}, \ldots)$$

with $k$-many alternation of quantifiers and a quantifier-free formula $G_A$ as a $k$-turn game in which:

- The first player begins by choosing the moves $\vec{y_1} \leq \vec{p_1}(\vec{x})$ altogether, then the second player chooses the moves $\vec{z_1} \leq \vec{q_1}(\vec{x})$ and they continue alternately.

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- The first player begins by choosing the moves $\vec{y_1} \leq \vec{p_1}(\vec{x})$ altogether, then the second player chooses the moves $\vec{z_1} \leq \vec{q_1}(\vec{x})$ and they continue alternately.
- if $G_A(\vec{x}, \vec{y_1}, \vec{z_1}, \vec{y_2}, \vec{z_2}, \ldots)$ becomes true the second player wins and otherwise the first player is the winner.

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The bridge between logic and games is:

**A Game Theoretic Characterization of Truth**

$A$ holds iff the second player has a *weak winning strategy*, meaning a function that reads all moves before $\vec{z_j}$ and computes a winning move $\vec{z_j}$.

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Note that in this equivalence, there is no need for the winning strategy to be an easy function. It is called *winning strategy* iff the functions are terms in the language and it is called $\mathcal{B}$-provable iff the fact that the strategy is a winning strategy is provable in the theory $\mathcal{B}$.

# Logic and Games

It is also possible to lift this equivalence to implications: Let $A, B$ are two bounded formulas in the form

$$A = \forall \vec{y}_1 \leq \vec{t}_1(\vec{x}) \exists \vec{z}_2 \leq \vec{s}_2(\vec{x}) \ldots G_A(\vec{x}, \vec{y}_1, \vec{z}_1 \ldots)$$

and

$$B = \forall \vec{u}_1 \leq \vec{p}_1(\vec{x}) \exists \vec{v}_2 \leq \vec{q}_2(\vec{x}) \ldots G_B(\vec{x}, \vec{u}_1, \vec{v}_1 \ldots)$$

Then we say $B$ is *weakly reducible* to $A$ iff there exists a sequence of functions, alternately reading the universal variables of $B$ to witness the universal quantifiers in $A$ and then reading the existential variables in $A$ to witness the existential variables in $B$.

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**A Game Theoretic Characterization of Implication**

$A \to B$ is valid iff $B$ is weakly reducible to $A$.

We call it *reducible* if all functions in the reduction are terms in the language and it is called $\mathcal{B}$-provable iff the fact that the reduction is a reduction between winning strategies is provable in the theory $\mathcal{B}$.

# The Main Theorem (informal)

Now we are ready to state our main witnessing theorem:

## The Main Theorem

Let $A(\vec{x})$ be a formula with $k$-many alternations of bounded quantifiers and $\mathcal{B}$ is the suitable base theory. Then $A(\vec{x})$ is provable via induction on the same kind of formulas iff there exists a uniform sequence of $k$-turn games $G_0, \ldots, G_{t(\vec{x})}$ for some term $t$ such that

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- An explicit winning strategy for the second player in $G_0$, provably in $\mathcal{B}$,
- a uniform sequence $W_i$ of explicit reduction from $G_{i+1}$ to $G_i$, provably in $\mathcal{B}$,
- and one explicit reduction from $A$ to $G_{t(\vec{x})}$, provably in $\mathcal{B}$.

In other words, $A$ is provable in such a theory iff the second player has a weak winning strategy in its corresponding game, constructed by iterating a $\mathcal{B}$-provable reduction, term many times.

# The language

## Definition *(the language)*

Let $\mathcal{L}$ be a first order language of arithmetic extending

$$\{0, 1, +, \dot{-}, \cdot, \lfloor \frac{\cdot}{\cdot} \rfloor, \leq\}$$

By $\mathcal{R}$ we mean the first order theory consisting of the axioms of commutative discrete ordered semirings (the usual axioms of commutative rings minus the existence of additive inverse plus the axioms to state that $\leq$ is a total discrete order such that $<$ is compatible with addition and multiplication with non-zero elements), plus the following defining axioms for $\dot{-}$ and $\lfloor \frac{\cdot}{\cdot} \rfloor$:

$$(x \geq y \to (x \dot{-} y) + y = x) \wedge (x < y \to x \dot{-} y = 0)$$

$$((y+1) \cdot \lfloor \frac{x}{y+1} \rfloor \leq x) \wedge (x \dot{-} (y+1) \cdot \lfloor \frac{x}{y+1} \rfloor < y+1)$$

# Complexity Classes

**Definition** *(the hierarchy)*

The hierarchy $\{\Sigma_k, \Pi_k\}_{k=0}^{\infty}$ is defined recursively in the following way:

(*i*) $\Pi_0 = \Sigma_0$ is the class of all quantifier-free formulas,

(*ii*) $\Sigma_k \subseteq \Sigma_{k+1}$ and $\Pi_k \subseteq \Pi_{k+1}$,

(*iv*) If $B(x) \in \Sigma_k$ then $\exists x \leq t\ B(x) \in \Sigma_k$ and $\forall x \leq t\ B(x) \in \Pi_{k+1}$ and

(*v*) If $B(x) \in \Pi_k$ then $\forall x \leq t\ B(x) \in \Pi_k$ and $\exists x \leq t\ B(x) \in \Sigma_{k+1}$.

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**Example**

Define $\mathcal{L}$ as the language consisting of all poly-time functions as function symbols. Then $\Sigma_k$ is the strict version of $\Sigma_k^b(\mathrm{PV})$ which characterizes the $k$-th level of the polytime hierarchy, $\Sigma_k^p$.

# Bounded Arithmetic

## Definition *(the theory)*

Let $\mathcal{A} \supseteq \mathcal{R}$ be a set of quantifier-free axioms and $\Phi$ be a class of bounded formulas closed under substitution and subformulas. By the first order bounded arithmetic, $\mathfrak{B}(\Phi, \mathcal{A})$ we mean the theory in the language $\mathcal{L}$ which consists of axioms $\mathcal{A}$, and the $\Phi$-induction axiom, i.e.,

$$A(0) \wedge \forall x(A(x) \rightarrow A(x+1)) \rightarrow \forall x A(x),$$

where $A \in \Phi$.

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## Example

With our definition of bounded arithmetic, different kinds of theories can be considered as bounded theories of arithmetic, for instance $IE_k$, $IU_k$, $T^k_n$, $I\Delta_0(\mathrm{exp})$ and $\mathrm{PRA}$ are just some of the well-known examples.

# Reductions

Let $\alpha \in \{\sigma, \pi\}$ and $A(\vec{x})$ and $B(\vec{x})$ be some bounded formulas in $\Pi_k$, $\{\vec{F}_i\}_{i=1}^k$ be a sequence of sequences of terms and $\mathcal{B} \supseteq \mathcal{R}$ a theory. By recursion on $k$, we will define $F = \{\vec{F}_i\}_{i=1}^k$ as an $\alpha$-reduction, from $B(\vec{x})$ to $A(\vec{x})$ and we will denote it by $A(\vec{x}) \geq_{\alpha}^{\mathcal{B},F} B(\vec{x})$ when:

(*i*) If $A, B$ are quantifier-free, a sequence of sequences of terms is both a $\sigma$-deterministic and a $\pi$-deterministic reduction from $B$ to $A$ iff $\mathcal{B} \vdash A(\vec{x}) \to B(\vec{x})$.

(*ii*) If $\alpha = \pi$, we have $A = \forall \vec{u} C(\vec{x}, \vec{u})$, $B = \forall \vec{v} D(\vec{x}, \vec{v})$ where the universal quantifiers are the whole block of left-most universal quantifiers (possibly empty) and $F = \{\vec{F}_i\}_{i=1}^{k+1}$ is a sequence of terms, then $A(\vec{x}) \geq_{\pi}^{\mathcal{B},F} B(\vec{x})$ iff

$$C(\vec{x}, \vec{F}_{k+1}(\vec{x}, \vec{v})) \geq_{\sigma}^{\mathcal{B},\hat{F}} D(\vec{x}, \vec{v})$$

where $\hat{F} = \{\vec{F}_i\}_{i=1}^{k}$.

# Reductions

(*iii*) If $\alpha = \sigma$, we have $A = \exists \vec{u} C(\vec{x}, \vec{u})$, $B = \exists \vec{v} D(\vec{x}, \vec{v})$ where the existential quantifiers are the whole block of left-most existential quantifiers (possibly empty) and $F = \{\vec{F}_i\}_{i=1}^{k+1}$ is a sequence of terms, then $A(\vec{x}) \geq_{\sigma}^{\mathcal{B},F} B(\vec{x})$ iff

$$C(\vec{x}, \vec{u}) \geq_{\pi}^{\mathcal{B},\hat{F}} D(\vec{x}, \vec{F}_{k+1}(\vec{x}, \vec{u}))$$

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where $\hat{F} = \{\vec{F}_i\}_{i=1}^{k}$.

We say $B$ is $\pi$-reducible to $A$ provably in $\mathcal{B}$ and we write $A \geq_{\pi}^{\mathcal{B}} B$, when there exists a sequence of sequences of terms $F$ such that $A \geq_{\pi}^{\mathcal{B},F} B$. We define $\sigma$-reducibility by replacing $\pi$ to $\sigma$ everywhere. Note that whenever the theory $\mathcal{B}$ is clear from the context, we drop it from the superscripts everywhere.

# Main Theorem (formal)

The following theorem characterizes the bounded consequences of the bounded theories of arithmetic by a uniform term-length sequence of reductions:

## Theorem

*Let* $A(\vec{x}) \in \Pi_k$*. Then* $\mathfrak{B}(\Pi_k, \mathcal{B}) \vdash A(\vec{x})$ *iff there exists a formula* $H(u, \vec{x}) \in \Pi_k$*, a* $\mathcal{B}$*-provable* $\pi$*-winning strategy for* $H(0, \vec{x})$*, a* $\mathcal{B}$*-provable* $\pi$*-reduction from* $H(u+1, \vec{x})$ *to* $H(u, \vec{x})$ *for* $u \leq t(\vec{x}) \dot{-} 1$ *for some term* $t$ *and a* $\mathcal{B}$*-provable* $\pi$*-reduction from* $A$ *to* $H(t(\vec{x}), \vec{x})$*.*

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Applying this characterization on Buss' hierarchy of bounded theories:

**Corollary**

Let $A(\vec{x}) \in \hat{\Pi}^b_k(\mathrm{PV})$. Then $T^k_2(\mathrm{PV}) \vdash A(\vec{x})$ iff the second player has a weak winning strategy in its corresponding game, constructed by iterating a $\mathrm{PV}$-provable polytime reduction, $2^{p(|\vec{x}|)}$ many times.

## Definition

Let $\mathcal{L} \supseteq \mathcal{L}_{\mathcal{R}}$ be a language. An instance of the $(j, k)$-game induction principle, $GI^j_k(\mathcal{L})$, is given by size parameters $a$ and $b$, a quantifier-free formula $G(u, \vec{v})$ with a fixed partition of the variables $\vec{v}$ into $k$ groups (interpreted as the $u$-th game $G_u(\vec{v})$ on moves $\vec{v}$), a sequence of terms $V$ and a uniform sequence $W_u$ of sequences of terms. The instance $GI(G, V, W, a, b)$ states that, interpreting $G(u, \vec{v})$ as a $k$-turn game on moves $\vec{v}$ in which all moves are bounded by $b$, the following cannot all be true:

- $(i)$ Deciding the winner of the game $G(0, \vec{v})$ depends only on the first $j$ moves,
- $(ii)$ The second player has a winning strategy for $G(0, \vec{v})$ (expressed as a $\Pi_j$ formula.)
- $(iii)$ For $u \leq a \dot{-} 2$, $W_u$ gives a $\pi$-reduction from $G(u+1, \vec{v})$ to $G(u, \vec{v})$,
- $(iv)$ $V$ is an explicit winning strategy for the first player in $G(a \dot{-} 1, \vec{v})$. (A $\pi$-reduction from $\bot$ to $G(a \dot{-} 1, \vec{v})$).

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- Secondly, observe that the complexity of the true formula is $\forall \Sigma_j$.
- Thirdly, note that proving its truth needs $\Pi_k$-induction, available in $\mathfrak{B}(\Pi_k, \mathcal{B})$.

# Another Characterization

The following theorem characterizes the bounded consequences of the bounded theories of arithmetic by game induction principle:

## Corollary

Let $j \leq k$ and $\mathcal{B}$ be a universal theory. Then:

$$\forall \Sigma_j [\mathfrak{B}(\Pi_k, \mathcal{B})] \equiv_{\mathcal{B}} GI_k^j(\mathcal{L}).$$

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It means that:

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- All formulas in $GI_k^j(\mathcal{L})$ are both in the form $\forall\Sigma_j$ and provable in $\mathfrak{B}(\Pi_k, \mathcal{B})$.
- For any $A \in \forall\Sigma_j$ that is provable in $\mathfrak{B}(\Pi_k, \mathcal{B})$, there exists $B \in GI_k^j(\mathcal{L})$ such that $A$ is $\pi$-reducible to $B$, provably in $\mathcal{B}$.

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## Corollary ([ST], [T])

For all $j \leq k$, $\forall\Sigma_j(T_2^k) \equiv_{\mathrm{PV}} GI_k^j(\mathcal{L}_{\mathrm{PV}})$.

Thank you for your attention!